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Application of an inner expansion method to plane, inviscid, compressible flow stability studies

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The inviscid compressible flow stability problem is mathematically similar to that of sound propagation in a sheared flow field. This similarity has been exploited by applying an inner expansion technique to study the effect of finite shear gradients on free parallel flow instabilities. This technique had previously been used to investigate the effect of thin boundary layers on sound propagation in ducts. The expansion, which is applicable to flow profiles involving thin, but finite, shear layers separating regions of uniform flow, offers a significant computational advantage over the numerical methods commonly employed to determine the stability of continuous mean flow profiles. Although equally applicable to three-dimensional and to spatially growing hydrodynamic instabilities, the procedure is demonstrated by application to the eigenvalue problem for temporal instabilities of shear layers and jets in plane inviscid compressible flow.

For the case of vanishingly thin shear layers, the eigenvalue equations derived here reduce to those obtained by Miles (1958) for parallel flows bounded by vortex sheets. The series solution of Graham & Graham (1969), valid for linear shear-layer profiles of arbitrary thickness, provides a basis of comparison for the expansion-method results. Unstable-mode eigenvalues obtained using the two methods are found to be in good agreement for a significant range of values of the ratio of shear-layer thickness to axial wavelength.

1. Introduction

Both temporal and spatial instabilities of jets and free layers in inviscid compressible flow have been studied with regard to their physical relationship to certain acoustic phenomena. These include turbulence-generated aerodynamic noise (Berman & Ffowcs Williams 1970; Crow & Champagne 1971; Michalke 1971) and the flow dependence of the acoustic impedance of orifices in duct walls (Ronneberger 1972). However, quite apart from physical considerations, there is a close connexion between the mathematical formulation of the two problems, a point demonstrated by the work of Miles (1957), Friedland & Pierce (1969) and Howe (1970). As noted by Doak (1972), the equation which governs the propagation of sound in lined ducts containing sheared mean flow also governs the hydrodynamic stability of the mean flow. In fact, several investigators have obtained extraneous solutions to the acoustic eigenvalue problem which are physically inexplicable from an acoustic viewpoint, but which can apparently be viewed as modes of hydrodynamic instability.

This mathematical similarity may be used to advantage by interchanging analytical and computational techniques between the two areas of study. The present work is a case in point, and involves application of an inner asymptotic expansion method to the instability problem. The method was used by Eversman & Beckemeyer (1972) in their investigation of boundary-layer effects on sound propagation in lined flow ducts.

The expansion technique is based on the assumption that a thin shear gradient separates areas of uniform flow. This allows use of the shear-layer thickness as a small parameter with which an inner asymptotic expansion in the shear gradient region may be performed. An approximate solution describing the behaviour of the disturbance pressure within the shear layer is thereby obtained, and subsequently matched to the uniform flow solutions by requiring that the boundary data coincide on the junctions between regions.

Significant computational advantages over the commonly employed numerical integration methods were obtained when this expansion procedure was applied to the acoustic problem, and these carry over to the present application. The method has provided a useful tool for the quantitative study of finite shear-gradient effects on jet and shear-layer instabilities.

2. Basic equations

Miles (1958) and Lessen, Fox & Zien (1965) have considered the temporal instability of discontinuous shear layers and jets in plane inviscid compressible flow, with the boundaries between uniform flow regions represented by vortex sheets. Blumen (1970, 1971), on the other hand, investigated the stability characteristics of smoothly varying mean flow profiles, specifically, with hyperbolic-tangent and hyperbolic-secant representations for the shear layer and jet, respectively. It is proposed that a profile intermediate to these be formed by replacing the vortex-sheet boundary between uniform flow regions by a shear layer of finite extent, as in figure 1. For convenience, we assume the flow profile to be continuous and the medium to be characterized by a constant sound speed a_0. The instability problem may be posed as a three-region coupled boundary-value problem, with propagation of the disturbance governed by an appropriate wave equation in each region (see figure 2). In region I we have

$$\nabla^2\overline{p} = (1/a_0)^2\overline{p}_{tt} + (2M_0/a_0)\overline{p}_{xt} + M_0^2\overline{p}_{xx}, \qquad (1)$$

while in region II

$$\nabla^2\overline{p} = (1/a_0)^2\overline{p}_{tt} + (2M/a_0)\overline{p}_{xt} + M^2\overline{p}_{xx} - 2\rho_0 a_0 M_y \overline{v}_x \qquad (2)$$

and in region III

$$\nabla^2\overline{p} = (1/a_0)^2\overline{p}_{tt}, \qquad (3)$$

where ρ_0 is the steady-state fluid density and $\overline{p}$ is the disturbance pressure. The velocity components are

$$u^* = U(y) + \overline{u}, \quad v^* = \overline{v}, \qquad (4)$$

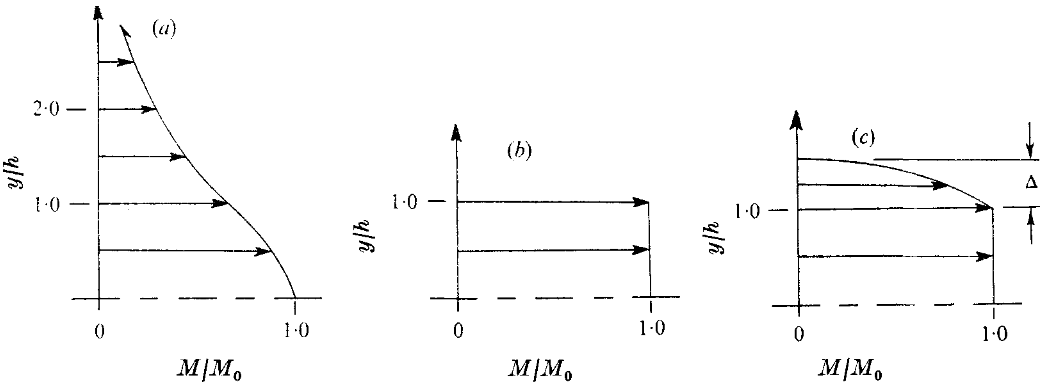

FIGURE 1. Jet profiles: (a) hyperbolic secant; (b) discontinuous vortex sheet; (c) finite shear layer.

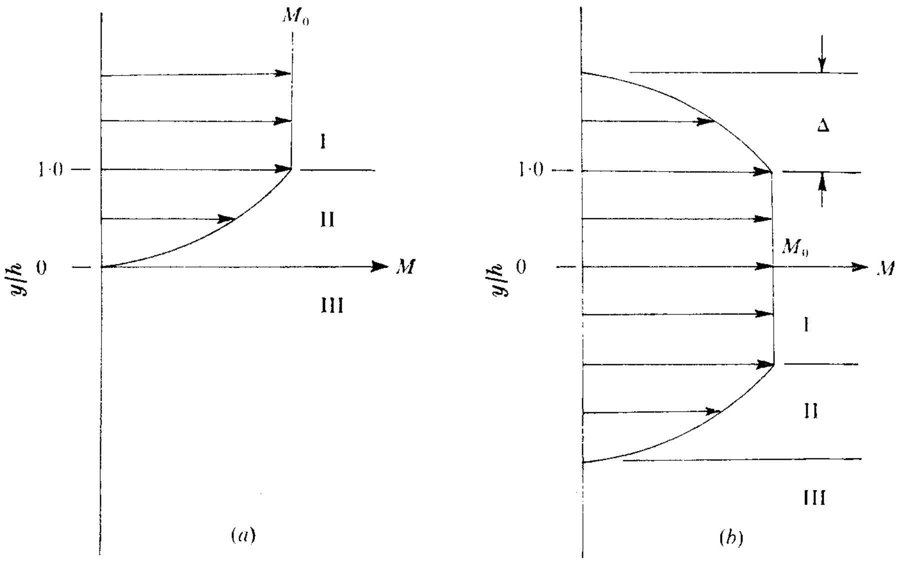

FIGURE 2. Flow regions: (a) shear layer; (b) jet.

where $\bar{u}$ and $\bar{v}$ are the disturbance velocities and U is the mean flow velocity. Note that

$$M = M_0 = U_0/a_0 \tag{5}$$

in region I,

$$M = M(y) = U(y)/a_0 \tag{6}$$

in region II and

$$M = U/a_0 = 0 \tag{7}$$

in region III.

Since the mean flow profile is continuous, continuity of the disturbance field may be assured by requiring continuity of pressure and pressure gradient $\partial\bar{p}/\partial y$ at the boundaries between regions. The boundary conditions may be completed by (i) requiring the disturbance to die out far from the shear layer or jet, and by (ii) invoking symmetry considerations at the jet centre-line.

We shall consider disturbances having wavenumber α, wave propagation velocity β_R/a_0 and temporal growth rate β_I/a_0, that is, the disturbance quantities vary as

$$\bar{g} = g(y)\exp[i(\alpha x - \beta t)], \tag{8}$$

where α is real,

$$\beta = \beta_R + i\beta_I \tag{9}$$

is complex and $\bar{g}$ stands for $\bar{p}$, $\bar{u}$ or $\bar{v}$. We note in passing that the problem formulation is not significantly changed for the study of spatial instabilities, in which case β becomes real, and α the complex eigenvalue. However, we restrict the present work to the study of temporal instabilities. The y momentum equation

$$-\rho_0(\bar{v}_t + U\bar{v}_x) = \bar{p}_y \tag{10}$$

may be used to relate the disturbance vertical velocity component and pressure in region II.

$$\bar{v}_x = i\alpha\bar{v} = \alpha\bar{p}_y/[\rho_0 a_0(\beta/a_0 - M\alpha)]. \tag{11}$$

Equations (1)–(3) may now be written as ordinary differential equations. In region I we have

$$d^2p/dy^2 + [(\alpha M_0 - \beta/a_0)^2 - \alpha^2]p = 0, \tag{12}$$

in region II

$$\frac{d^2p}{dy^2} - \frac{2\alpha}{\alpha M - \beta/a_0}\frac{dM}{dy}\frac{dp}{dy} + [(\alpha M - \beta/a_0)^2 - \alpha^2]p = 0 \tag{13}$$

and in region III

$$d^2p/dy^2 + [(\beta/a_0)^2 - \alpha^2]p = 0. \tag{14}$$

Solutions in regions I and III may be written down immediately. For the shear-layer problem, with the regions defined such that (figure 2)

$$\infty > y \geqslant \Delta \tag{15}$$

in region I,

$$\Delta \geqslant y \geqslant 0 \tag{16}$$

in region II and

$$0 \geqslant y > -\infty \tag{17}$$

in region III, the solutions are

$$p = A\exp\{i[(\alpha M_0 - \beta/a_0)^2 - \alpha^2]^{\frac{1}{2}}y\} \tag{18}$$

in region I and

$$p = B\exp\{i[(\beta/a_0)^2 - \alpha^2]^{\frac{1}{2}}\, y\} \tag{19}$$

in region III, where the signs of the square-root terms are chosen so that the imaginary part is positive in region I and negative in region III, thus ensuring that the disturbance dies out far from the shear layer.

Similarly, for the jet, where

$$0 \geqslant y \geqslant -h \tag{20}$$

defines the extent of region I,

$$-h \geqslant y \geqslant -(h+\Delta) \tag{21}$$

for region II and

$$-(h+\Delta) \geqslant y > -\infty \tag{22}$$

for region III, the disturbance pressure is

$$p = A\,{}^{\cos}_{\sin}\{[(\alpha M_0 - \beta/a_0)^2 - \alpha^2]^{\frac{1}{2}}y\} \tag{23}$$

in region I, with the cosine chosen for disturbances symmetric in pressure (antisymmetric in v) about the jet centre-line and the sine for disturbances antisymmetric in pressure (symmetric in v). The solution in region III is

$$p = B\exp\{i[(\beta/a_0)^2 - \alpha^2]^{\frac{1}{2}}(y + h + \Delta)\}, \tag{24}$$

with the imaginary part of the square root negative. We are left only with the tasks of determining the solutions in region II and then satisfying the continuity conditions for the disturbance field. Both numerical and closed-form solutions to (13) have been obtained previously. Graham & Graham (1969) determined a series solution for the special case of a linear shear-layer profile. Numerical solutions have been obtained, for example, by numerical integration (Mungar & Plumblee 1969) and by weighted residual techniques (Unruh 1972). We choose to assume the shear-layer region to be of small but finite thickness, and to use this assumption to obtain an approximate closed-form solution.

3. The inner expansion

Make the change of variables

$$\zeta = y \tag{25}$$

for the shear layer and

$$\zeta = y+h+\Delta \tag{26}$$

for the jet and, furthermore, specify values for the pressure and its gradient at $\zeta = 0$. Then the disturbance pressure in region II is governed by the following equations:

$$\frac{d^2p}{d\zeta^2}-\frac{2\alpha}{\alpha M-\beta/a_0}\frac{dM}{d\zeta}\frac{dp}{d\zeta}+[(\alpha M-\beta/a_0)^2-\alpha^2]p = 0, \tag{27}$$

$$p = \pi_0, \quad dp/d\zeta = \pi_1 \quad \text{at} \quad \zeta = 0. \tag{28}$$

We assume the shear-layer thickness Δ to be very small and construct an expansion of the sheared region by introducing the inner variable

$$\eta = \zeta/\Delta. \tag{29}$$

Note that we have transformed the boundary-value problem for region II into an initial-value problem and have focused our attention on the area near $\eta = 0$.

The governing equations may now be written as

$$\frac{d^2p}{d\eta^2}-\frac{2\alpha}{\alpha M-\beta/a_0}\frac{dM}{d\eta}\frac{dp}{d\eta}+\Delta^2[(\alpha M-\beta a_0)^2-\alpha^2]p = 0, \tag{30}$$

$$p = \pi_0, \quad dp/d\eta = \Delta\pi_1 \quad \text{at} \quad \eta = 0. \tag{31}$$

We proceed by assuming the solution of (30) and (31) to be in the form of a power series in Δ

$$p(\eta) = p_0(\eta)+\Delta p_1(\eta)+\Delta^2 p_2(\eta)+\dots. \tag{32}$$

By substituting (32) into (30) and (31), and equating like powers of Δ, we obtain a succession of initial-value problems. Specifically, for the zeroth-order term we obtain

$$\frac{d^2p_0}{d\eta^2}-\frac{2\alpha}{\alpha M-\beta/a_0}\frac{dM}{d\eta}\frac{dp_0}{d\eta} = 0, \tag{33}$$

$$p_0 = \pi_0, \quad dp_0/d\eta = 0 \quad \text{at} \quad \eta = 0, \tag{34}$$

for the first-order term

$$\frac{d^2p_1}{d\eta^2} - \frac{2\alpha}{\alpha M - \beta/a_0}\frac{dM}{d\eta}\frac{dp_1}{d\eta} = 0, \tag{35}$$

$$p_1 = 0, \quad dp_1/d\eta = \pi_1 \quad \text{at} \quad \eta = 0, \tag{36}$$

and for the remaining terms

$$\frac{d^2p_j}{d\eta^2} - \frac{2\alpha}{\alpha M - \beta/a_0}\frac{dM}{d\eta}\frac{dp_j}{d\eta} = [\alpha^2 - (\alpha M - \beta/a_0)^2]\,p_{j-2}, \tag{37}$$

$$p_j = 0, \quad dp_j/d\eta = 0 \quad \text{at} \quad \eta = 0 \quad (j = 2, 3, \ldots). \tag{38}$$

Recognizing the expression

$$1/(\alpha M - \beta/a_0)^2 \tag{39}$$

as an integrating factor for each of (33), (35) and (37), we see that the solutions up to the second-order term are

$$p_0 = \pi_0, \tag{40}$$

$$p_1 = \frac{\pi_1}{(\beta/a_0)^2}\int_0^\eta (\alpha M - \beta/a_0)^2\,d\sigma, \tag{41}$$

$$p_2 = -\pi_0\int_0^\eta \left\{(\alpha M - \beta/a_0)^2\left[\mu + \alpha^2\int_0^\mu \frac{d\sigma}{(\alpha M - \beta/a_0)^2}\right]\right\}d\mu. \tag{42}$$

Upon neglecting terms of order Δ^2, we may express the disturbance pressure and its derivative in region II as

$$p = \pi_0 + \frac{\pi_1\Delta}{(\beta/a_0)^2}\int_0^\eta (\alpha M - \beta/a_0)^2\,d\sigma, \tag{43}$$

$$\frac{dp}{dy} = \frac{\pi_1(\alpha M - \beta/a_0)^2}{(\beta/a_0)^2} - \pi_0\Delta(\alpha M - \beta/a_0)^2\left[\eta - \alpha^2\int_0^\eta \frac{d\sigma}{(\alpha M - \beta/a_0)^2}\right]. \tag{44}$$

Although this solution is strictly valid only for small η, we now assume it to hold as well in the outer region near the opposite edge of the shear layer $\eta = 1$. Then the pressure and its Neumann derivatives at the boundaries of region II may be represented by

$$p(\eta = 1) = \pi_0 + \frac{\pi_1\Delta}{(\beta/a_0)^2}\int_0^1 (\alpha M - \beta/a_0)^2\,d\sigma, \tag{45}$$

$$\frac{dp}{dy}(\eta = 1) = \frac{\pi_1(\alpha M_0 - \beta/a_0)^2}{(\beta/\alpha_0)^2} - \pi_0\Delta(\alpha M_0 - \beta/a_0)^2\left[1 - \alpha^2\int_0^1 \frac{d\sigma}{(\alpha M - \beta/a_0)^2}\right] \tag{46}$$

and $$p(\eta = 0) = \pi_0, \quad \frac{dp}{dy}(\eta = 0) = \pi_1. \tag{47}$$

The limits of applicability for this approximate solution must be determined numerically. Note that the solution depends on the form of the mean flow profile within the shear layer in a relatively straightforward fashion. The integrals involved can be solved in closed form for several profiles, including sinusoidal, linear and the $1/N$ power law profiles.

To investigate the accuracy of the solution, attention was given to the linear profile. For this case,

$$M = M_0\eta \tag{48}$$

and the integrals in (45) and (46) are

$$\int_0^1 (\alpha M - \beta/a_0)^2 \, d\sigma = \tfrac{1}{3}(\alpha M_0)^2 - \alpha M_0 \, \beta/a_0 + (\beta/a_0)^2 \tag{49}$$

and
$$\int_0^1 \frac{d\sigma}{(\alpha M - \beta/a_0)^2} = \frac{-1}{(\beta/a_0)(\alpha M_0 - \beta/a_0)}. \tag{50}$$

A basis of comparison for the expansion-method solution is offered by the exact series solution to (13) for the linear profile. This solution, which is not restricted to small Δ, was obtained by Graham & Graham (1969) in their investigation of plane wave propagation through a shear layer of linear profile. It may be written, in our notation, as

$$p = c_0 f + c_1 g, \tag{51}$$

where c_0 and c_1 are arbitrary constants, and

$$f = \sum_{j=1}^{\infty} b_j [1 - M_0 \alpha\eta/(\beta/a_0)]^{j-1} \quad (j \text{ odd}), \tag{52}$$

$$g = \sum_{j=2}^{\infty} b_j [1 - M_0 \, \alpha\eta/(\beta/a_0)]^{j-1} \quad (j \text{ even}). \tag{53}$$

The coefficients b_j are given by the recursive relationships

$$b_1 = b_4 = 1, \quad b_2 = 0, \quad b_3 = -\tfrac{1}{2}(\Delta\beta/a_0 \, M_0)^2, \tag{54}$$

$$b_j = \left[\frac{\Delta(\beta/a_0)}{M_0 \alpha}\right]^2 \left[\frac{\alpha^2 b_{j-2} - (\beta/a_0)^2 b_{j-4}}{(j-1)(j-4)}\right] \quad (j = 5, 6, \ldots). \tag{55}$$

The disturbance pressures in region II defined by (43)–(47) and by (51)–(55), respectively, have been matched to the solutions in regions I and III to yield closed-form eigenvalue equations for the temporal instability problem. Before discussing the numerical techniques used to solve the transcendental characteristic equations, let us make some observations pertaining to the eigenvalue equations for the thin-shear-layer approximation.

By requiring field continuity at the junctions between regions, we obtain a set of four homogeneous algebraic equations in the four unknowns, A, B, π_0 and π_1. We seek a non-trivial solution to these equations and, thus, require the determinant of the coefficients of the unknowns to vanish. Expansion of the determinant yields the following characteristic equations:

$$(\beta/a_0)^2 (G_0^2 - \alpha^2)^{\frac{1}{2}} - G_0^2 F = -\Delta \left\{ F (G_0^2 - \alpha^2)^{\frac{1}{2}} \int_0^1 G^2 d\sigma - i (G_0 \beta/a_0)^2 \left[1 - \alpha^2 \int_0^1 G^{-2} d\sigma \right] \right\} \tag{56}$$

for the shear layer and

$$\mp{}^{\tan}_{\cot}[h(G_0^2 - \alpha^2)^{\frac{1}{2}}] = \left\{ \frac{-iFG_0^2 + \Delta(\beta/a_0)^2 G_0^2 \left[1 - \alpha^2 \int_0^1 G^{-2} d\sigma\right]}{(\beta/a_0)^2 (G_0^2 - \alpha^2)^{\frac{1}{2}} + i\Delta F (G_0^2 - \alpha^2)^{\frac{1}{2}} \int_0^1 G^2 \, d\sigma} \right\} \tag{57}$$

for the jet, where

$$G^2 = (\alpha M - \beta/a_0)^2, \; G_0^2 = (\alpha M_0 - \beta/a_0)^2, \; F = [(\beta/a_0)^2 - \alpha^2]^{\frac{1}{2}}. \tag{58}$$

Note that, in the limit as $\Delta \to 0$, the eigenvalue equations reduce to

$$(\beta/a_0)^2 (G_0^2-\alpha^2)^{\frac{1}{2}} - [(\beta/a_0)^2-\alpha^2]^{\frac{1}{2}} G_0^2 = 0 \tag{59}$$

for the shear layer and

$$\begin{matrix}\tan \\ -\cot\end{matrix} [h(G_0^2-\alpha^2)^{\frac{1}{2}}] = \frac{i[(\beta/a_0)^2-\alpha^2]^{\frac{1}{2}} G_0^2}{(\beta/a_0)^2 (G_0^2-\alpha^2)^{\frac{1}{2}}}, \tag{60}$$

or

$$\begin{matrix}\coth \\ \tanh\end{matrix} [ih(G_0^2-\alpha^2)^{\frac{1}{2}}] = \frac{(\beta/a_0)^2 (G_0^2-\alpha^2)^{\frac{1}{2}}}{G_0^2 [(\beta/a_0)^2-\alpha^2]^{\frac{1}{2}}} \tag{61}$$

for the jet. These are identical to the results originally given by Miles (1958).

The principal advantage of the expansion-method solution is quite evident from the above equations. We observe that the presence of the finite shear layer appears as a perturbation to the idealized vortex-sheet eigenvalue equations. The added terms make the equations somewhat complex for a detailed qualitative study of shear-layer effects. However, the perturbed equations are ideally suited to an iterative numerical solution approach in which the vortex-sheet eigenvalues may be used as initial guesses for the finite shear-layer solutions.

4. Numerical results

The algorithm used in solving the transcendental eigenvalue equations for both the series and the expansion methods involves solution of a series of equations, each for an increasingly thicker shear layer. The eigenvalues obtained at each state are used as initial guesses in a Newton–Raphson iteration for the next thickness. This procedure is repeated until the desired thickness is attained. Initial values for the discontinuous profile for the examples reported here were taken from the work of Miles (1958) and Lessen *et al.* (1965).

Because the eigenvalue equations are available in closed form, their derivatives with respect to the eigenvalue, required in the Newton–Raphson approach, were performed analytically and evaluated directly. Computer programs incorporating these solution procedures have been written in FORTRAN IV. Computations were performed on an IBM 360/65, using single-precision complex arithmetic.

Figure 3 illustrates the accuracy of the expansion method relative to the series solution for the case of a jet bounded by a linear shear layer. The expansion method provides excellent values of the temporal growth rate for shear-layer thickness to axial wavelength ratios ($\alpha\Delta$) up to 0·25. For higher $\alpha\Delta$ values, the growth rate is overestimated, with the error approaching 10–15% at $\alpha\Delta = 0{\cdot}5$.

Figures 4(*a*) and (*b*) illustrate the effects of a finite linear shear layer on the variation with wavelength of the wave speed and temporal growth rate of jet instabilities at Mach 1·0. Figures 4(*a*) and (*b*) relate to instabilities antisymmetric in pressure (symmetric in v) and symmetric in pressure (antisymmetric in v), respectively. In addition, the results of Blumen (1971) for a hyperbolic-secant jet profile at Mach 1·0 are superimposed on figures 4 (*a*) and (*b*) for comparison. Generally, agreement between the expansion-method and series solutions is quite good, with wave speeds being equal to within graphical accuracies and the

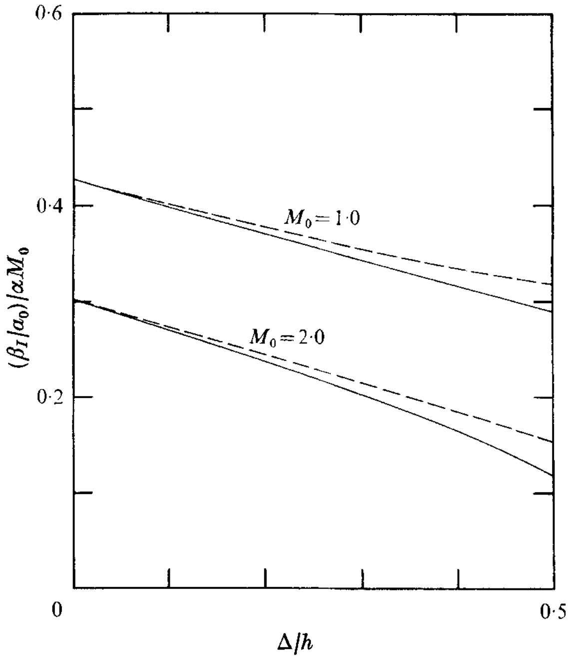

FIGURE 3. Jet bounded by linear shear layer. Temporal growth rate *vs.* shear-layer thickness. Disturbance wavelength = 1·0. ———, series solution; — — —, expansion method.

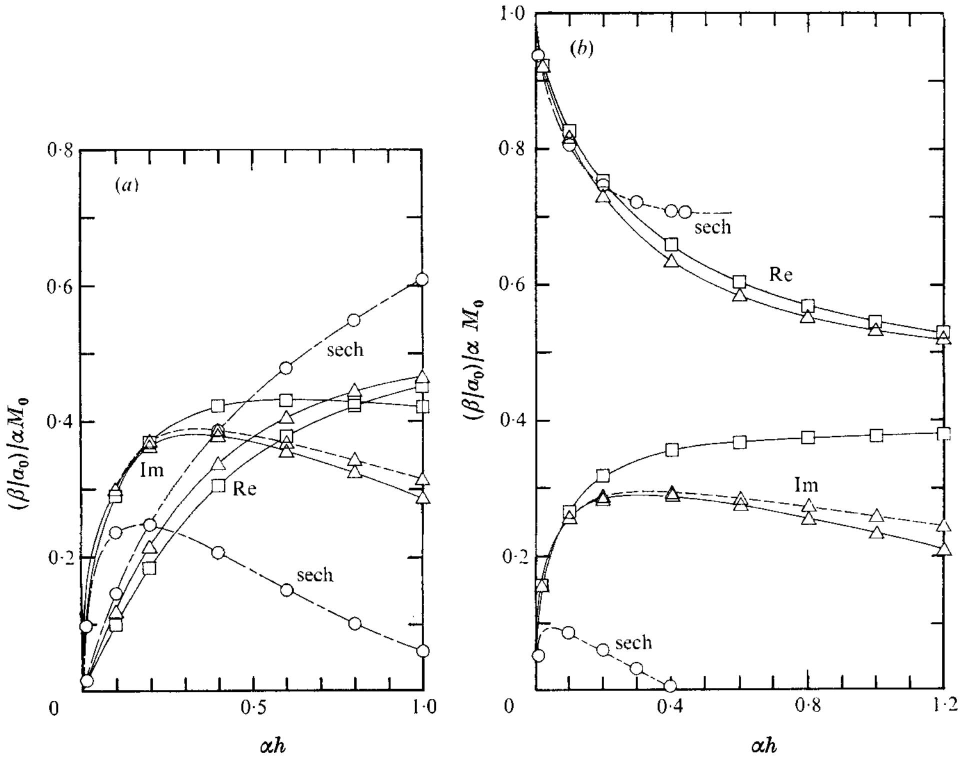

FIGURE 4. Jet bounded by thin shear layer. Temporal growth rate $(\beta_I/a_0)/\alpha M_0$ and wave speed $(\beta_R/a_0)/\sigma\alpha M_0$ *vs.* ratio of jet half-height to disturbance wavelength. (*a*) Unstable disturbance antisymmetric in pressure. (*b*) Unstable disturbance symmetric in pressure. $M_0 = 1·0$. □, linear, $\Delta = 0·025\,h$; △, linear, $\Delta = 0·500\,h$; ——, series solution; – – –, expansion method.

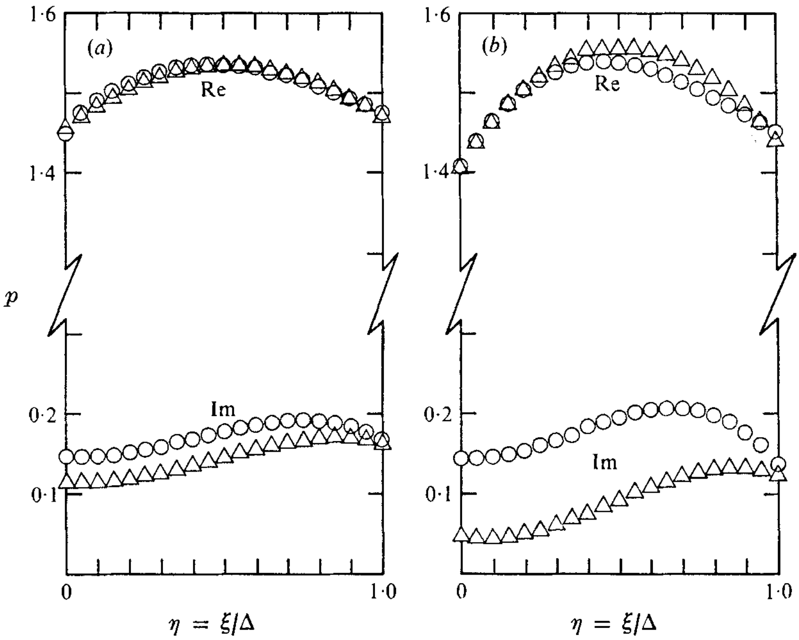

FIGURE 5. Disturbance pressure distribution in linear-profile shear layer. Unstable anti-symmetric pressure disturbance. Wavelength = 1·0. (a) Δ/h = 0·25. (b) Δ/h = 0·50, M_0 = 1·0. △, series solution; ○, expansion method.

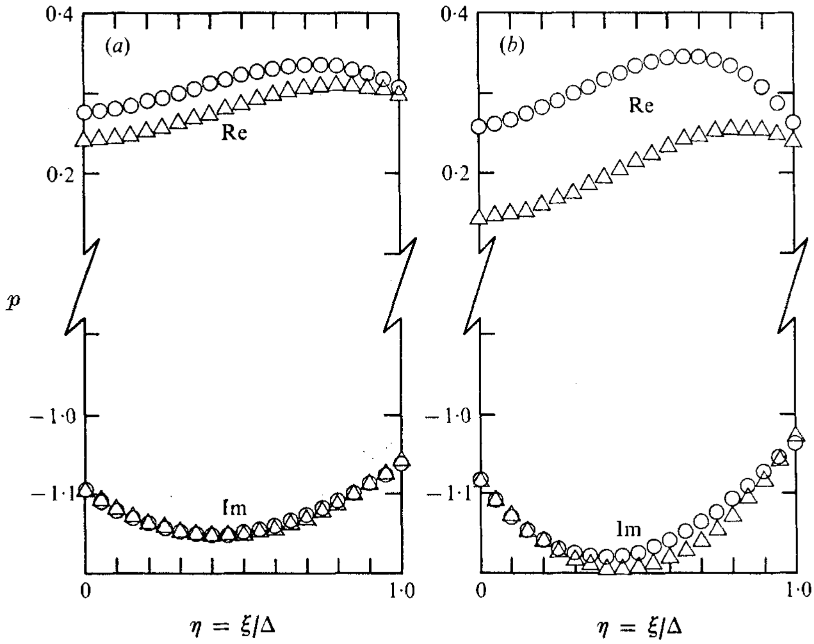

FIGURE 6. Disturbance pressure distribution in linear-profile shear layer. Unstable symmetric pressure disturbances. Wavelength = 1·0. (a) Δ/h = 0·25. (b) Δ/h = 0·50, M_0 = 1·0. △, series solution; ○, expansion method.

temporal growth rate being slightly overestimated. The expansion procedure accurately predicts the trends with increasing wavelength for the ranges of parameters investigated. Initial guesses for the results shown in these figures were taken from figures 4 and 6 of Lessen *et al.* (1965).

Figures 5 and 6 illustrate the pressure distributions through the shear layer, as predicted by the expansion and series solutions, for antisymmetric and symmetric pressure modes, respectively. Each figure relates to cases with unit values of Mach number and wavelength, and shear-layer thicknesses $\Delta/h = 0{\cdot}25$ and $0{\cdot}50$. The distributions have been normalized so that the coefficient A of the pressure in region I [equations (18) and (23)] equals $1{\cdot}0 + 0{\cdot}0i$. Once again, correlation between the two methods is excellent at $\Delta/h = 0{\cdot}25$, with a significant loss of accuracy at $\Delta/h = 0{\cdot}50$, although the general form of the mode shape is approximated rather well in both cases.

The studies reported here were primarily intended to verify the expansion method and to delineate the limits of its applicability. Additional investigations were conducted for more complicated shear-layer profiles. Specifically considered were sinusoidal and square-root Mach number profiles for which the integrals in (45) and (46) could be evaluated in closed form. Values of the wave speeds and growth rates predicted for the various linear and nonlinear profiles, as well as additional comparisons of the type detailed above, are given elsewhere (Beckemeyer 1973*a*).

5. Conclusions

This expansion method provides a useful tool for the quantitative study of shear-gradient effects on jet and shear-layer instabilities. Available results for discontinuous vortex-sheet representations may be easily extended, using the expansion-method approach, to cases involving finite shear layers. The simple numerical techniques involved offer considerable computational savings over the numerical integration methods commonly employed in continuous flow profile stability investigations.

Although illustrated by application to the problem of temporal instabilities in plane inviscid compressible flow, the method is equally applicable to the study of spatial and three-dimensional instabilities. Extension to these problems, as well as to the case with flow and temperature gradients, is currently under investigation (Beckemeyer 1973*b*). For the two-dimensional examples considered here, the method has been shown to yield accurate results for growth rates and wave speeds of unstable disturbances for shear-layer thickness to disturbance wavelength ratios up to $0{\cdot}25$, and to provide valid trend indications for even higher values.

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